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OPTIMAL TRAJECTORIES IN REPRODUCTION MODELS OF ECONOMIC DYNAMICS

S.I. HAMIDOV[1]

Abstract. A reproduction model with Leontief type production functions is considered. The production mapping of such model is usually given by superlinear multivalued mappings. We consider effective model trajectories, and we use lossless equilibrium mechanism to construct such trajectories. We find the characteristics of effective trajectories and we show that the consumption problem reaches its maximum on effective trajectories. We also find the conditions under which it is impossible to construct an effective trajectory using lossless equilibrium mechanism.

Keywords: reproduction model, lossless equilibrium mechanisms, characteristics.

AMS Subject Classification: C61, C62, C65, D50.

1. Introduction

The study of the behavior of trajectories of any economic dynamics model is an important problem, both for economists and mathematicians. Many authors have dealt with this problem [1, 2, 4, 12, 15-19]. This paper also studies the behavior of effective trajectories of some economic dynamics models.

We consider Neumann type model of economic dynamics Z [4, 8, 12-14, 19]. As it is known, such model is given by superlinear multivalued mappings. The states of this model are the vectors in $(R_+^n)^n$ with non-negative coordinates. We consider Leontief type production functions [4, 10, 11]. Effective trajectories have a property that they admit a characteristic. We consider the effective trajectories whose characteristic prices are positive. Using the concept of lossless equilibrium mechanism, we find a formula for calculating the components of effective trajectory. We prove a theorem that provides the conditions under which it is impossible to construct an effective trajectory with the use of lossless equilibrium mechanism whatever the initial state is. The concept of lossless equilibrium mechanism has been first introduced by A.M. Rubinov [3, 5, 6, 8, 9, 20].

Let us consider the model Z. Note that the states for this model for all t are the vectors $X \in \left(R_+^n\right)^n$, and its production mapping a_t has a form [13, 15, 18]

$$a_t\left(X\right) = \left\{Y = \left(y^{1\cdot},\ \ldots,\ y^{n\cdot}\right) \in \left(R_+^n\right)^n | 0 \leq Y \leq \left(BF\right)_t\left(X\right)\right\},$$

where

$$\left(BF\right)_t\left(X\right) = \sum_{k=1}^{n} B_t^k \cdot x^{k\cdot} + \left(F_t^1\left(x^{1\cdot}\right),\ \ldots,\ F_t^n\left(x^{n\cdot}\right)\right),$$

$$X = \left(x^{1\cdot},\ \ldots,\ x^{n\cdot}\right) \in \left(R_+^n\right)^n.$$

[1]Baku State University, Baku, Azerrbaijan
e-mail: sabir818@yahoo.com
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Here B_t^k are $n \times n$ non-negative matrices, F_t^k is a production function of the $k-$th industry [10, 18]. This function is defined by the equality

$$F_t^k\left(x^{k\cdot}\right) = \min_{i=\overline{1,\ n}} \frac{x^{ki}}{c_t^{ik}}\ , \qquad x^{k\cdot} = \left(x^{k1},\ \ldots,\ x^{kn}\right) \in R_+^n. \tag{1}$$

Production mapping a_t^k of the industry k at the moment t has a form

$$a_t^k\left(x\right) = \left\langle 0,\ B_t^k \cdot x + \left(0,\ \ldots,\ 0,\ F_t^k\left(x\right),\ 0,\ \ldots,\ 0\right)\right\rangle,\ \ x \in R_+^n.$$

Let $X_t = \left(x_t^{1\cdot},\ \ldots,\ x_t^{n\cdot}\right)$ be an effective trajectory of this model. This trajectory admits a characteristic $L_t = \left(\ell_t^{1\cdot},\ \ldots,\ \ell_t^{n\cdot}\right)$. Here $\ell_t^{i\cdot} \in R_+^n$. The coordinates of the vector $\ell_t^{i\cdot}$ stand for the price of the products in the $i-$th industry at the moment t. As showed A.M. Rubinov [22] by$t > 0$ and $\ell_t^{i\cdot} \gg 0\left(i = \overline{1,\ n}\right)$ the prices of the products do not depend on the industry they belong to, i.e., we can assume $\ell_t^{i\cdot} = \ell_t$for all t. In other words,

$$L_t = \left(\ell_t,\ \ldots,\ \ell_t\right),$$

where $\ell_t = \left(\ell_t^1,\ \ldots,\ \ell_t^n\right) \in R_+^n$.

We will consider only those effective trajectories, whose characteristic prices for all products are positive, i.e.

$$\ell_t^i > 0 \qquad \text{for all} \qquad i,\ t,$$

where ℓ_t^i is a price of the $i-$th product at the moment t.

2. Main results

Introduce the functions

$$U_t^k\left(\ell_{t+1},\ x\right) = \left[\ell_{t+1},\ B_t^k \cdot x\right] + \ell_{t+1}^k \cdot F_t^k\left(x\right). \tag{2}$$

Recall that the relation

$$\mu_t^k\left(x\right) = \frac{U_t^k\left(\ell_{t+1},\ x\right)}{\left[\ell_t,\ x\right]}\left(k \in I\right), \tag{3}$$

is a growth rate of the total wealth of the $k-$th industry in the state $x$with the prices ℓ_{t+1} and ℓ_t.

Lemma 2.1. Let the trajectory $\left(X_t\right)_{t=1}^{\infty}$ of the model Z process for all t the following properties: the vector X_t is constructed as a part of equilibrium in the model $M = \left(\{y\},\ U_t\left(\ell_{t+1}\right),\ \ \Omega_t,\ V\right)$, where $V = \left(R_+^n,\ \ldots,\ R_+^n\right), y = \left(BF\right)_{t-1}\left(X_{t-1}\right)$, and the vector of budgets $\Omega_t = \left(\lambda_t^1,\ \ldots,\ \lambda_t^n\right)$ is chosen such that e is $\mu_t^k = 1$, Besides the equilibrium prices coincide with $\ell_t = \left(\ell_t^1,\ \ldots,\ \ell_t^n\right)$, and the budgets λ_t^k are related to ℓ_t by the formula $\lambda_t^k = \left[\ell_t,\ x_t^{k\cdot}\right]$. Then $L_t = \left(\ell_t,\ \ldots,\ \ell_t\right)$ is characteristic of the trajectory $\left(X_t\right)$.

Proof. B*y the definition of characteristic for the trajectory* $\left(X_t\right)$, we have

$$\left[L_t,\ X_t\right] = \left[L_{t+1},\ X_{t+1}\right], \qquad \forall\ t > 0,$$

a*nd for every trajectory* $\left(\tilde{X}_t\right)$ *of the model* Z *the inequalities*

$$\left[L_t,\ \tilde{X}_t\right] = \left[L_{t+1},\ \tilde{X}_{t+1}\right], \qquad \forall\ t > 0,$$

hold.

By the conditions of the Lemma 2.1. the vectors $x_t^{k\cdot}$, forming the state of the trajectory X_t have a property that the functions $U_t^k(\ell_{t+1},\ x)/[\ell_t,\ x]$ reaches it maximum on them i.e.

$$\max_{x\geq 0}\frac{U_t^k(\ell_{t+1},\ x)}{[\ell_t,\ x]}=\frac{U_t^k\left(\ell_{t+1},\ x_t^{k\cdot}\right)}{\left[\ell_t,\ x_t^{k\cdot}\right]}=1,\qquad \left(k=\overline{1,\ n}\right). \tag{4}$$

Consider $[L_{t+1},\ X_{t+1}]$ taking into account that $x_{t+1}^{k\cdot}\in a_t^k\left(x_t^{k\cdot}\right)\left(k=\overline{1,\ n}\right).$

$$[L_{t+1},\ X_{t+1}]=\sum_{k=1}^{n}\left[\ell_{t+1},\ x_{t+1}^{k\cdot}\right]=\sum_{k=1}^{n}\left(\left[\ell_{t+1},\ B_t^k\cdot x_t^{k\cdot}+\ell_{t+1}^k\cdot F_t^k\left(x_t^{k\cdot}\right)\right]\right)=$$
$$=\sum_{k=1}^{n}U_t^k\left(\ell_{t+1},\ x_t^{k\cdot}\right).$$

Using the formula (3) we get

$$[L_{t+1},\ X_{t+1}]=\sum_{k=1}^{n}U_t^k\left(\ell_{t+1},\ x_t^{k\cdot}\right)=\sum_{k=1}^{n}\left[\ell_t,\ x_t^{k\cdot}\right]=[L_t,\ X_t].$$

Then

$$[L_{t+1},\ X_{t+1}]=[L_t,\ X_t],$$

and for any other trajectory $\tilde{X}_t$, we have

$$\left[L_{t+1},\ \tilde{X}_{t+1}\right]=\sum_{k=1}^{n}\left[\ell_{t+1},\ \tilde{x}_{t+1}^{k\cdot}\right]=\sum_{k=1}^{n}\left(\left[\ell_{t+1},\ B_t^k\cdot\tilde{x}_t^{k\cdot}\right]+\ell_{t+1}^k\cdot F_t^k\left(\tilde{x}_t^{k\cdot}\right)\right)=$$
$$=\sum_{k=1}^{n}U_t^k\left(\ell_{t+1},\ \tilde{x}_t^{k\cdot}\right).$$

But

$$\frac{U_t^k\left(\ell_{t+1},\ x_t^{k\cdot}\right)}{\left[\ell_t,\ x_t^{k\cdot}\right]}=\max_{x\geq 0}\frac{U_t^k(\ell_{t+1},\ x)}{[\ell_t,\ x]}=1,\qquad \left(k=\overline{1,\ n}\right).$$

Consequently

$$\left[L_{t+1},\ \tilde{X}_{t+1}\right]\leq\sum_{k=1}^{n}U_t^k\left(\ell_{t+1},\ \tilde{x}_t^{k\cdot}\right)=\sum_{k=1}^{n}\left[\ell_t,\ \tilde{x}_t^{k\cdot}\right]=\left[L_t,\ \tilde{X}_t\right],$$

i.e.

$$\left[L_t,\ \tilde{X}_t\right]\geq[L_{t+1},\ X_{t+1}],\quad \forall\ t\geq 0.$$

So we have shown that $L_t=(\ell_t,\ \ldots,\ \ell_t)$ is a characteristic for the trajectory (X_t) of the model Z.

From (4) it follow that for all k

$$\max_{x\geq 0}\frac{U_t^k(\ell_{t+1},\ x)}{[\ell_t,\ x]}=\frac{U_t^k\left(\ell_{t+1},\ x_t^{k\cdot}\right)}{\left[\ell_t,\ x_t^{k\cdot}\right]}=\mu_t^k=1. \tag{5}$$

Note that in this case, the maximum values of the growth rate of the total wealth of all sectors $k=\overline{1,\ n}$ coincide with each other and are equal to 1.

Consider the vector $\overline{x}_t^{k\cdot}=\left(\overline{x}_t^{k1},\ \ldots,\ \overline{x}_t^{kn}\right)$. Let $\overline{x}_t^{k\cdot}$ be some solutions of the following consumer problem

$$U_t^k\left(\ell_{t+1},\ x\right) \rightarrow max \quad \text{subject to} \quad \left[\ell_t,\ x\right] = 1, \quad x \geq 0. \tag{6}$$

Introduce the set of indexes

$$I = \{1,\ 2,\ \ldots,\ n\},$$

$$R\left(\overline{x}_t^{k\cdot}\right) = \left\{ i \in I \mid \frac{\overline{x}_t^{ki}}{c_t^{ik}} = \min_{j \in I} \frac{x_t^{kj}}{c_t^{jk}} \right\} (k \in I), \quad t = 1,\ 2,\ \ldots. \tag{7}$$

Then if $R\left(\overline{x}_t^{k\cdot}\right) \neq I$ then

$$\frac{\overline{x}_t^{ki}}{c_t^{ik}} < \frac{x_t^{kj}}{c_t^{jk}} \qquad \text{for} \quad \forall\ i \in R\left(\overline{x}_t^{k\cdot}\right), \qquad j \in I \backslash R\left(\overline{x}_t^{k\cdot}\right).$$

This means that for the given $\overline{x}_t^{k1},\ \ldots,\ \overline{x}_t^{kn}$ the number of the product in the industry k at the moment t, involved in the production process is not $x_t^{kj}\left(j \in I \backslash R\left(x_t^{k\cdot}\right)\right)$, is $\frac{c_t^{jk}}{c_t^{ik}} \cdot \overline{x}_t^{ki}$ where $\frac{c_t^{jk}}{c_t^{ik}} \cdot \overline{x}_t^{ki} < \overline{x}_t^{kj}$, and $\overline{x}_t^{kj} - \frac{c_t^{jk}}{c_t^{ik}} \cdot \overline{x}_t^{ki}$ is the number of products not really involved in the production. This quantity will be called a loss at the moment t.

In what follows, we will consider only those trajectories on which the above situation is impossible, i.e.

$$R\left(\overline{x}_t^{k\cdot}\right) = I\left(k \in I, \quad t = 1,\ 2,\ \ldots\right). \tag{8}$$

Now, let us consider the effectiveness of the trajectories of the model Z, admitting the characteristic (L_t).

Let $(X_t)_{t=1}^{\infty}$ be an effective trajectory of the model Z with characteristics (L_t). Then the vectors $x_t^{k\cdot}$ forming the state of the effective trajectory X_t may be re presented as

$$x_t^{k\cdot} = \lambda_t^k \cdot \overline{x}_t^{k\cdot}\,(k \in I), \tag{9}$$

where $\lambda_t^k = \left[\ell_t,\ \overline{x}_t^{k\cdot}\right]$ is a positive number, $\overline{x}_t^{k\cdot}$ is a solution of the consumer problem (6).

Let (8) hold. Consider the matrix $C_t = \left(c_t^{ij}\right)_{i,\ j=1}^{n}$ and column vectors of the matrix C_t

$$c_t^{\cdot k} = \begin{pmatrix} c_t^{1k} \\ c_t^{2k} \\ \vdots \\ c_t^{nk} \end{pmatrix}, (k \in I). \tag{10}$$

Throughout this we will assume that the determinant of the matrix $C_t\,(t = 1,\ 2,\ \ldots)$ is not zero. Then, with given budgets $\lambda_t^k\,(k \in I)$ and prices ℓ_t it follows from (7), (8) and (9) that the vectors x_t^k forming the state of the effective trajectoryX_t of the model Z i.e. vectors which are the lossless equilibrium states of the model M with fixed budgets, are calculated by the formula:

$$x_t^{k\cdot} = \frac{\lambda_t^i}{\left[\ell_t,\ c_t^{\cdot k}\right]} \cdot c_t^{\cdot k}\,(k \in I). \tag{11}$$

Now, let's show how to find ℓ_{t+1} for the given ℓ_t. Suppose that the vector $\ell_t = \left(\ell_t^1,\ \ldots,\ \ell_t^n\right)$ and the state $X_t = \left(x_t^{1\cdot},\ \ldots,\ x_t^{n\cdot}\right)$ of the economy at the moment t are known. By this state we define the vector of resources $y = \left(y^1,\ \ldots,\ y^n\right)$ distributed at the moment $t+1$ where

$$y^i = \sum_{k\in I} \nu_t^{ki} \cdot x_t^{ki} + F_t^i \left(x_t^{i\cdot}\right) (i \in I)\,, \tag{12}$$

and the vector $x_t^{k\cdot}$ is defined by the formula (11). Substituting (11) into (1) we obtain

$$F_t^i \left(x_t^{i\cdot}\right) = \frac{\lambda_t^i}{\left[\ell_t,\ c_t^{\cdot i}\right]} (i \in I)\,.$$

Denote

$$\beta_t^i = \frac{\lambda_t^i}{\left[\ell_t,\ c_t^{\cdot i}\right]} (i \in I)\,, \quad t = 1,\ 2,\ \ldots \tag{13}$$

and consider the vector

$$\beta_t = \left(\beta_t^1,\ \ldots,\ \beta_t^n\right), \quad t = 1,\ 2,\ \ldots\ .$$

Taking into account the above considerations from (12) we obtain

$$y = (E + C_t^\nu) \cdot \beta_t\ , \tag{14}$$

where

$$E = diag\ \ (1,\ \ldots,\ 1)\,, \quad C_t^\nu = \left(\hat{c}_t^{ij}\right)_{i,\ j=1}^n\ ,$$

$$\hat{c}_t^{ij} = \nu_t^{ji} \cdot c_t^{ij} \left(i,\ j = \overline{1,\ n};\quad t = 1,\ 2,\ \ldots\right). \tag{15}$$

By (8), X_{t+1} is a lossless equilibrium state [9] which must satisfy to the balance equality

$$\sum_{k=1}^n x_{t+1}^{k\cdot} = y,$$

i.e. using (11) for the moment $t+1$ we get

$$\sum_{k=1}^n \frac{\lambda_{t+1}^k}{\left[\ell_{t+1},\ c_{t+1}^{\cdot k}\right]} \cdot c_{t+1}^{\cdot k} = y,$$

or

$$C_{t+1} \cdot \beta_{t+1} = y. \tag{16}$$

From (14) and (16) it follows that

$$C_{t+1} \cdot \beta_{t+1} = (E + C_t^\nu) \cdot \beta_t.$$

Hence,

$$\beta_{t+1} = C_{t+1}^{-1} \cdot (E + C_t^\nu) \cdot \beta_t, \qquad t = 1,\ 2,\ \ldots\ . \tag{17}$$

Rewriting the formula (17) in coordinate form and substituting (13), we obtain a system of n scalar equations in n variables - the coordinates of the vector ℓ_{t+1}.

Remark 2.1. The characteristics (L_t) (here $L_t = (\ell_t,\ \ldots,\ \ell_t)$ of trajectory (X_t) is constructed inductively using the formula (17), with $\ell_{t+1} \gg 0$ and the vectors $x_t^{k\cdot}$, wich form the state X_t of this trajectory, are defined by the formula (11).

Remark 2.2. If the sequence $\left(\tilde{L}_t\right)$, constructed inductively using (17), is not a characteristics for the trajectory (X_t), for initial vector ℓ, then for some t either the inequality

$$(BF)_t\,(X_t) = \sum_{i=1}^n x_{t+1}^{i\cdot} = \sum_{i=1}^n \lambda_{t+1}^i \cdot \overline{x}_{t+1}^{i\cdot}\ , \quad t = 1,\ 2,\ \ldots, \tag{18}$$

does not hold or the condition

$$\ell_{t+1} \gg 0, \quad t = 1,\ 2,\ \ldots .$$

is not satisfied.

Remark 2.3. Recurrence formula (17) can be obtained from the system of equations (4), wich is based on the equality of the maximum growth rates of the total wealth in different industries.

The equilibrium mechanism in the case of $R\left(\overline{x}_t^{k\cdot}\right) = I\,(k \in I, \quad t = 1,\ 2,\ \ldots)$ is called a lossless equilibrium mechanism. Let's construction effective trajectory (X_t) of the model Z using the lossless equilibrium mechanism. Since the state of the effective trajectory are determined by the vector β_t, we must to define this vector. To do this, we introduce the following notations:

$$\begin{array}{c} \tilde{E}_t = E + C_t^{\nu}\ , \\ \tilde{C}_t = \tilde{E}_t^{-1} \cdot C_{t+1}\ . \end{array} \quad t = 1,\ 2,\ \ldots \tag{19}$$

Then the formula (17) becomes

$$\beta_{t+1} = \tilde{C}_t^{-1} \cdot \beta_t\ .$$

Note that the lossless equilibrium mechanism is realizable when $\beta_t > 0\,(t = 1,\ 2,\ \ldots)$ which is equivalent to the validity of the equalities (18) for all $k \in I$. In fact, in the case where $\beta_t^k = 0$ for some $k \in I$, in order for the states $x_t^{i\cdot}$ to maximize the growth rate of the total wealth $\left(\gamma_t^i\right)$ in the corresponding industries, and for these growth rates to be equal at some moment, we have to consider the inequalities instead of (18)

$$\sum_{i \in I} x_{t+1}^{i\cdot} \le (BF)_t\,(X_t)\,,$$

i.e. the lossless equilibrium mechanism is broken.

It is therefore of interest to know: is the case where the lossless equilibrium mechanism is broken possible?

Consider the case where the matrices C_t coincide for all t with the same matrix $C = \left(c^{ij}\right)_{i,\ j=1}^{n}$. Recall that the number $c^{i,j} > 0\,(i,\ j \in I)$ indicates the quantity of the product of the $i-$th industry used in the production of unit product of the $j-$th industry.

Let

$$\nu_t^{ij} = 0 \qquad \forall\ i \in I, \quad j > 1, \quad t = 1,\ 2,\ \ldots,$$
$$\nu_t^{i} = \nu^{i1} = \nu^{i}, \qquad t = 1,\ 2,\ \ldots, \quad i \in I.$$

This means that all the resources in the production process, except the first one, have been completely used up. We can assume that there is only one fund forming industry in the model Z.

We will say that trajectory $(X_t)_{t=1}^{\infty}$ can be constructed in the model Z the using the lossless equilibrium mechanism if the vectors $x_t^{k\cdot}$ forming the state of the given trajectory X_t lie in the conical hull stretched on the vectors of type $c^{\cdot k}\,(k \in I)$ at any time t.Let $\overline{X}$ be a Niemann equilibrium vector.

Theorem 2.1. *Let there be only one fund-forming industry in the model Z and the numbers $\nu^i\,(i \in I)$ are such that*

$$\nu^i < \frac{c^{1i}}{\sum\limits_{k=2}^{n} c^{1k} \cdot c^{ki}}\ .$$

Then, no effective trajectory $(X_t)_{t=1}^{\infty}$ can be constructed using the lossless equilibrium mechanism what ever the initial state $X_1 \neq \lambda \overline{X}\,(\lambda \geq 0)$ is (19)

Proof. It is obvious that in this case from (15), (19) *we have*

$$\tilde{E}_t = \tilde{E} = \left(\varepsilon^{ij}\right)_{i,\ j=1}^{n}, \qquad \varepsilon^{ij} = \begin{cases} 1+\nu^1 \cdot c^{11} & \text{by} \quad i=j=1, \\ \nu^j \cdot c^{1j} & \text{by} \quad i=1, \quad j>1, \\ 1 & \text{by} \quad i=j=\overline{2,\ n}\ , \\ 0 & \text{by} \quad i \neq j, \quad i \neq 1. \end{cases}$$

Then

$$\tilde{E}^{-1} = \left(\widetilde{\varepsilon}^{ij}\right)_{i,\ j=1}^{n}, \qquad \widetilde{\varepsilon}^{ij} = \begin{cases} \dfrac{1}{1+\nu^1 \cdot c^{11}} & \text{by} \quad i=j=1, \\ -\dfrac{\nu^j \cdot c^{1j}}{1+\nu^1 \cdot c^{11}} & \text{by} \quad i=1, \quad j>1, \\ 1 & \text{by} \quad i=j=\overline{2,\ n}\ , \\ 0 & \text{by} \quad i \neq j, \quad i \neq 1. \end{cases}$$

Hence,

$$\widetilde{N} = \tilde{E}^{-1} \cdot N = \left(\widetilde{n}^{ij}\right)_{i,\ j=1}^{n},$$

$$\widetilde{n}^{ij} = \begin{cases} \dfrac{c^{1j} - \sum\limits_{k=2}^{n} \nu^k \cdot c^{1k} \cdot c^{kj}}{1+\nu^1 \cdot c^{11}} & \text{by} \quad i=1; \\ \\ c^{ij} by \quad i \neq j. \end{cases} \tag{20}$$

In our case, the formula (17) becomes

$$\beta_{t+1} = \tilde{C}^{-1} \cdot \beta_t\ , \quad t=1,\ 2,\ \ldots \tag{21}$$

or

$$\beta_{t+1} = \tilde{C}^{-t} \cdot \beta_t\ , \quad t=1,\ 2,\ \ldots. \tag{22}$$

It is known that [7] if $\tilde{C}$ is a positive matrix and $x \neq \lambda \cdot \overline{x}\,(\lambda \geq 0)$ (i.e. x is not an eigenvector of the matrix $\tilde{C}$) then there $t > 0$ such that $\tilde{C}^{-t} \cdot x$ is not positive. Let's find the condition under which the matrix $\tilde{C}$ is positive. From (20) we see that for this to take place, it is necessary that the following system of inequalities fold.

$$\begin{cases} c^{11} > \nu^2 \cdot c^{12} \cdot c^{21} + \cdots + \nu^n \cdot c^{1n} \cdot c^{n1}\ , \\ c^{12} > \nu^2 \cdot c^{12} \cdot c^{22} + \cdots + \nu^n \cdot c^{1n} \cdot c^{n2}\ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c^{1n} > \nu^2 \cdot c^{12} \cdot c^{2n} + \cdots + \nu^n \cdot c^{1n} \cdot c^{nn}, \\ \nu^1 \geq 0\ . \end{cases} \tag{23}$$

From this system, we obtain

$$\nu^i < \frac{1}{c^{ii}} \quad \text{for all} \quad i > 1. \tag{24}$$

Here c^{ii} is a quantity of the product i, necessary for the production of the unit product$i$ (i.e. itself). Then it is natural to assume $c^{ii} < 1$ (or$\frac{1}{c^{ii}} \gg 1$).Thus the condition (24) is natural in this model, because, $\nu^i \in [0,\ 1]\,(i \in I)$.

Let all $\nu^i\,(i \in I)$ differ little from each other, i.e. the quantity of the first product remaining in at the $i-$th $(i \in I)$ industry after production process is over differ little from other. This can be written as follows $\nu^i = \nu - \varepsilon_i$, where $\varepsilon_i \geq 0\ \ (i \in I)$.

Then the system (23) becomes

$$\begin{cases} c^{11} + \sum\limits_{i=2}^{n} \varepsilon_i \cdot c^{1i} \cdot c^{i1} > \nu \cdot \sum\limits_{i=2}^{n} c^{1i} \cdot c^{i1} \ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c^{1n} + \sum\limits_{i=2}^{n} \varepsilon_i \cdot c^{1i} \cdot c^{in} > \nu \cdot \sum\limits_{i=2}^{n} c^{1i} \cdot c^{in} \ , \\ \nu^1 \geq 0 \ . \end{cases}$$

Denote

$$\varepsilon = \max_{i \in I} \varepsilon_i \ .$$

Then the last system may be rewritten as follows:

$$\begin{cases} \nu - \varepsilon < \dfrac{c^{11}}{c^{12} \cdot c^{21} + \cdots + c^{1n} \cdot c^{n1}} \ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \nu - \varepsilon < \dfrac{c^{1n}}{c^{12} \cdot c^{2n} + \cdots + c^{1n} \cdot c^{nn}} \ , \\ \nu^1 \geq 0. \end{cases}$$

Note that all terms in denominators on the right-hand sides of the above inequalities are all of the second order of smallness (as $c^{ii} < 1,\ i, j \in I$), and all terms in numerators are all of the first order of smallness. That is, the conditions obtained are "natural" ones.

Thus, we have shown that the matrix C is positive, which means that for any vector β_1, which is not an eigenvector of the matrix C, there exists $t > 0$ such that β_{t+1} is not positive. Consequently, the lossless equilibrium mechanism does not allow constructing effective trajectory.

The theorem is proved.

Consequence. If $\nu^{ij} = 0\ \forall i, j \in I$, in the model Z, then, for $X_1 \neq \lambda X$ $(\lambda \geq 0)$, the lossless equilibrium mechanism does not allow one to construct an effective trajectory $(X_t)_{t+1}^{\infty}$.

3. Conclusion

The following results are obtained in this paper:

(1) The characteristics of the trajectories of the considered model are determined under some conditions (see Lemma 1);
(2) The efficiency of the trajectories of the considered model with characteristics is investigated;
(3) The conditions are found for the construction of effective trajectories using lossless equilibrium mechanisms.

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Sabir Hamidov was born in 1954. In 1976 he graduated from Baku State University, Faculty of Applied Mathematics and Cybernetics. In 1986, he got Ph.D. degree. Since 2000 he has been working as an associate professor at the Department "Mathematical Cybernetics" of Baku State University. His research interests include non-smooth optimization issues, discrete mathematics, mathematical logic and discrete mathematical models and economic dynamics models.